

FLUCTUATION BOUNDS FOR INTERFACE FREE ENERGIES IN SPIN GLASSES

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ABSTRACT. We consider the free energy difference restricted to a finite volume for certain pairs of incongruent thermodynamic states (if they exist) in the Edwards-Anderson Ising spin glass at nonzero temperature. We prove that the variance of this quantity with respect to the couplings grows proportionally to the volume in any dimension greater than or equal to two. As an illustration of potential applications, we use this result to restrict the possible structure of Gibbs states in two dimensions.

1. INTRODUCTION

The quantitative dependence of free energy fluctuations on the quenched disorder is a fundamental thermodynamic property of all disordered systems. Bounds on such fluctuations were used in [2] to prove uniqueness of the Gibbs state in a class of two-dimensional disordered systems, including the two-dimensional random field Ising model. In the case of a finite-volume spin glass with a specified boundary condition, the free energy is a random function of the couplings within the volume: as the couplings vary, so does the free energy. One may also consider for a fixed volume the *difference* in free energy between two different boundary conditions, say periodic and antiperiodic; this quantity is similarly a random function of the couplings. Because such a change in boundary conditions induces an interface between their respective finite-volume Gibbs states, one can refer equivalently to the dependence of the resulting *interface free energy* on the couplings. The interface between two spin configurations (in the present case, each chosen from a different Gibbs state corresponding to the two different boundary conditions) is defined as the set of couplings satisfied in one configuration and unsatisfied in the other.

It has been noted by several authors [13, 15, 16, 17, 21, 27, 29] that determining quantitative bounds on the interface free energy variance with respect to the couplings would allow a resolution of a long-standing open problem at the heart of the statistical mechanics of short-range spin glasses in finite dimensions: how many pure states (or at zero temperature, ground states) are present in the spin glass phase? To begin, one has to specify what one means by multiple states; this was elucidated nicely in [15] where it was noted that in the context of spin glasses it is particularly important to differentiate between *incongruent* and *regionally congruent* pure states. In the case of d -dimensional Ising spin glasses, the latter are related by a global spin flip everywhere except near interfaces whose dimensionality is

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strictly smaller than d ; otherwise, the states are incongruent. So, for example, incongruent *ground* states differ by a “space-filling” interface. Whether incongruent pure and ground states exist has been a subject of controversy for many years. One of the main goals in studying free energy fluctuations of the corresponding interfaces is to provide the tools for settling this question.

More generally, for a Gibbs state Γ , consider a correlation function $\Gamma(\sigma_{x_0}\sigma_{y_0})$ (often denoted $\langle\sigma_{x_0}\sigma_{y_0}\rangle_\Gamma$) for two fixed, nearest-neighbor sites x_0 and y_0 . Then multiple incongruent or regionally congruent states are present if there exists more than one thermodynamic limit for such a correlation function (along different subsequences of volumes), say $\Gamma(\sigma_{x_0}\sigma_{y_0})$ and $\Gamma'(\sigma_{x_0}\sigma_{y_0})$ [16]. Incongruent states are those in which such (x_0, y_0) constitute a nonvanishing fraction of edges:

Definition 1.1. *Two states Γ and Γ' are incongruent if for some $\varepsilon > 0$ there is a subset of edges (x_0, y_0) with strictly positive density such that $|\Gamma(\sigma_{x_0}\sigma_{y_0}) - \Gamma'(\sigma_{x_0}\sigma_{y_0})| > \varepsilon$.*

It is known [26] that if one uses coupling-*independent* boundary conditions (which include the usual ones of free, periodic, or fixed, chosen along a deterministic sequence of volumes) to generate distinct states, then these states must be incongruent. Because of this natural property, they were referred to in [27] as *observable* states.

In contrast, regionally congruent states, which differ by zero-density interfaces such as those found in homogeneous ferromagnets using Dobrushin boundary conditions, could also exist in spin glasses in principle, but if so could only be generated using coupling-*dependent* boundary conditions [26], using procedures as yet unknown. Such states, if they exist, would be of mathematical interest, but whether their presence would carry any physical or thermodynamical significance is at present unclear. In what follows we will concern ourselves exclusively with incongruent pure or ground states.

Several authors, using various techniques, have considered the problem of free energy fluctuations of a finite-volume Ising spin glass with a single boundary condition [5, 8, 9, 13]. In contrast, there are few results for the variance of free energy differences. Based on heuristic scaling arguments, it was proposed in [14] that, in general dimension, the variance of the fluctuations of such free energy differences is bounded, up to a constant, by the system’s surface area. This conjecture was proved for free energy differences between finite-volume Gibbs states generated by gauge-related boundary conditions such as periodic and antiperiodic [1, 20] (see also Sect. 5.7 in [10]). A key quantity that remains to be determined is therefore a *lower* bound for interface free energy fluctuations. Indeed, the various competing pictures of the low-temperature spin glass phase predict different size dependences of such fluctuations. Determining a lower bound, however, has remained elusive due to a number of problems which will be discussed below. In this paper we provide an initial step by obtaining such a lower bound, for the free energy difference within a finite volume for certain classes of distinct infinite-volume Gibbs states (cf. Assumption 4.1) assuming such classes exist. We note that the lower bound obtained here is for a quantity that differs somewhat from that for which an upper bound has been obtained, which prevents us from applying of our result to dimensions higher than two.

2. BASIC OUTLINE OF PROOF

Before proceeding with the technical details of the proof, we present a brief sketch of its main ideas and basic ingredients. We will always work in the periodic boundary condition metastate [2, 22, 23, 24], although our results can be extended to any metastate generated using coupling-independent boundary conditions (free, fixed, and so on). The technical definition of the metastate will be presented in Sect. 3, but for now a rough definition will suffice: the metastate can be thought of as a probability measure on all of the thermodynamic states that appear in an infinite sequence of volumes with specified boundary conditions.

One longstanding technical hurdle in studying free energy fluctuations for spin glass Hamiltonians is the problem of “keeping track” of the selected Γ and Γ' among the set of all possible Gibbs states as couplings vary. In contrast to Hamiltonians for ordered systems (such as ferromagnets), there is no known connection between spin glass Gibbs states and boundary conditions — or more precisely, none that can be made in a translation-invariant, measurable way (which will be needed later). There are several possible ways to address this issue. For example, if at least some of the Gibbs states have nonzero weight in the metastate, one can use these weights as state “identifiers”. A different way to solve the above issue is to randomly sample Γ and Γ' among the Gibbs states in the metastate(s). We will examine both of these possibilities in this paper.

A second hurdle concerns the “cancellation problem”: unlike the ferromagnet (including the random-field Ising model), the sign of the energy difference between two spin configurations varies with the couplings that constitute the interface. Consequently, although each edge will presumably contribute an energy fluctuation of order one, the fluctuations may cancel as one moves along the interface. This could in principle lead to an overall energy fluctuation that varies sublinearly with the volume, or may even be volume-independent. The latter may indeed be the case *a priori*, and indeed corresponds to some proposed scenarios of the spin glass phase. The problem is that, because of these cancellations, any of the usual techniques used to study fluctuations of free energy differences between incongruent states could lead to poor estimates of their volume dependence and therefore inconclusive results.

In order to get more accurate estimates of the volume dependence of the free energy difference, one needs to find a suitable quantity to study. There are in principle a number of different possible choices, all of which behave like the free energy difference between two states, and all of which should be physically indistinguishable; but most are not suitable for obtaining a lower bound. More specifically, previous attempts to obtain a good lower bound failed because the choice of a free energy difference-type quantity — typically, looking at the free energy difference in a finite volume with two different boundary conditions — interfered with the overall translation-covariance properties of the spin glass model in all space. Hence, one key feature of the argument in this paper relies on the choice of a quantity which, although it also is a finite-volume free energy difference, manages to preserve the overall translation-invariance properties of the infinite system. There are several possible choices; we will use a quantity similar to that used in [2] (see in particular Eq. (5.11)), which provides several technical advantages to alternative possibilities.

Preservation of translation-invariance is necessary but not sufficient to circumvent the cancellation problem. The presence of incongruence, along with the suitable choice of a

quantity to measure free energy differences between states, allows one to condition free energy differences on individual couplings in a translation-invariant way; this results in the desired random variable (for now, call it μ_{xy}) which reflects the extent of free energy difference fluctuations on the coupling J_{xy} , but whose distribution does not depend on the choice of bond (x, y) . The most straightforward way to arrive at such a quantity is to average over all couplings except one, but (in a manner similar to that described in the preceding paragraph) such averaging can lead to cancellations such that the variance of μ_{xy} tends to zero as the volume increases.

To address this issue, we adopt a modified approach, using blocks rather than individual bonds. That is, we now consider μ_{xy} to be conditioned on a *block* b centered at (x, y) . Because μ_{xy} as a function of the couplings is not a constant, the variance of μ_{xy} will not tend to zero for b sufficiently large; this follows from the martingale convergence theorem, as discussed in Sect. 5. Importantly, the size of b is independent of both (x, y) and Λ , because of the translation covariance of μ_{xy} . Consequently, in this approach different blocks of the large finite volume under study contribute equally to the variance, leading to a stronger lower bound on the growth of fluctuations, which indeed are found to grow as the square root of the volume.

With these considerations in mind, we now sketch the main steps of the proof. As noted, we work in the periodic boundary condition metastate, which has strong coupling- and translation-covariance properties built in. These covariance properties, along with the translation-invariance of the coupling distribution itself, naturally extend to measures defined on the couplings and/or the thermodynamic states on which the metastate is supported. These “nice” properties allow one to equate a (suitably defined) derivative of the free energy difference, with respect to a specific coupling J_{xy} , to $\Gamma(\sigma_x \sigma_y) - \Gamma'(\sigma_x \sigma_y)$, the latter being the difference in the indicated two-spin correlation function evaluated in Γ and Γ' , respectively. The assumption of incongruent states implies this difference will be of order one with positive probability in the couplings.

The next step is to construct a *martingale decomposition* [11] of the free energy difference. This involves dividing the volume Λ into blocks whose size is independent of Λ , but which are large enough so that the metastate average of $\Gamma(\sigma_x \sigma_y) - \Gamma'(\sigma_x \sigma_y)$, conditioned on the couplings in the block, is nonzero. We then consider a sequence of (metastate averages of) the free energy differences in Λ , with each element in the sequence conditioned on the couplings in an increasing number of blocks. The variance of the free energy difference over *all* the couplings in Λ is easily shown to be no smaller than the sum of the variances of the differences between the k^{th} and the $(k - 1)^{\text{th}}$ elements of the sequence, with k running from 1 to N , the number of blocks in the full volume.

The remainder of the proof uses the translation-covariant properties of the metastate to show that the variance of the k^{th} difference is independent of k . Because (by the assumption of existence of incongruence) the variance of the first block is of order one independent of the volume (this result doesn’t precisely follow from the incongruence assumption, but requires only a small amount of further work), the result follows.

In this paper we do not consider upper bounds of the same quantity in dimensions greater than two. In two dimensions, however, some results can be obtained by following the approach of [2] (see also [7] for a detailed exposition). They are based on the elementary fact that the free energy difference (in any dimension) is bounded uniformly by the sum of the

couplings on the boundary of Λ ; in two dimensions, one can arrive at a contradiction using the martingale central limit theorem, but following [2], it is sufficient to show that the values of the free energy difference rescaled by $|\Lambda|^{1/2}$ are unbounded. While the results are not surprising — one expects in fact a single pure thermodynamic state in two dimensions at all positive temperatures — they illustrate the potential power of the techniques described below to determine the structure of the low temperature spin glass phase.

3. PRELIMINARIES

We consider the Edwards-Anderson (EA) Hamiltonian [12] on a finite box $\Lambda = [-L, L]^d \subset \mathbb{Z}^d$

$$(1) \quad H_{\Lambda, J}(\sigma) = - \sum_{(x, y) \in E(\Lambda)} J_{xy} \sigma_x \sigma_y, \quad \sigma \in \{-1, 1\}^\Lambda.$$

For $B \subset \mathbb{Z}^d$, we write $E(B)$ for the set of edges with both ends in B . The couplings $J_\Lambda := (J_{xy}, (x, y) \in E(\Lambda))$ are i.i.d. ν -sampled random variables. (We will sometimes abuse notation and write ν for the joint distribution of the couplings as well as its marginals.) We assume throughout that $\nu(dJ_{xy})$ is continuous and that $\int \nu(dJ_{xy}) J_{xy}^4 < \infty$.

To proceed we need to introduce some notation. Write $\Sigma = \{-1, +1\}^{\mathbb{Z}^d}$ and let $\mathcal{M}_1(\Sigma)$ be the set of (regular Borel) probability measures on Σ . An infinite-volume Gibbs state Γ for the Hamiltonian (1) is an element of $\mathcal{M}_1(\Sigma)$ that satisfies the DLR equations [18] for that Hamiltonian. The Gibbs state Γ induces a probability measure on infinite-volume spin configurations: for a function $f(\sigma)$ on the spins

$$(2) \quad \Gamma(f(\sigma)) = \int d\Gamma f(\sigma).$$

The set of Gibbs states corresponding to the coupling realization J is denoted by \mathcal{G}_J . The inverse temperature β is fixed throughout.

We are now in a position to define the quantity we will be studying. In a finite box Λ and at inverse temperature $0 \leq \beta < \infty$, consider the *difference of free energies* between two infinite-volume Gibbs states Γ and Γ' for the Hamiltonian (1)

$$(3) \quad F_\Lambda(J, \Gamma, \Gamma') = \log \frac{\Gamma(\exp \beta H_{\Lambda, J}(\sigma))}{\Gamma'(\exp \beta H_{\Lambda, J}(\sigma'))}.$$

Note that $\Gamma(\exp \beta H_{\Lambda, J}(\sigma))$ is a ratio whose numerator comprises a Boltzmann factor on spin configurations only in Λ^c , the *complement* of Λ , while the denominator is the usual Boltzmann factor on all infinite-volume spin configurations. Consequently, $F_\Lambda(J, \Gamma, \Gamma')$ amounts to the difference in free energies within the volume Λ between boundary conditions chosen from the infinite-volume thermodynamic states Γ' and Γ .

For the coupling realization J , consider now two probability measures on $\mathcal{M}_1(\Sigma)$ that are supported on the set of Gibbs states; we will denote them κ_J and κ'_J . We will study the free energy fluctuations of two independently chosen states Γ and Γ' under the probability measure

$$(4) \quad M := \nu(dJ) \kappa_J(d\Gamma') \times \kappa'_J(d\Gamma).$$

(We will also consider below a case where Γ and Γ' are chosen from the same κ_J .) The most natural measures κ_J to consider are the ones that are obtained by taking (deterministic) subsequential limits of finite-volume Gibbs measures, as discussed in [2, 3, 22, 23, 24]. In so doing the κ_J 's will inherit two useful invariance properties from the finite-volume Gibbs measures.

The first of these is translation-covariance. Formally, let T be a translation of \mathbb{Z}^d , and consider the operation mapping $\mathcal{M}_1(\Sigma)$ to itself: $\Gamma \mapsto T\Gamma$, where

$$(5) \quad T\Gamma(f(\sigma)) := \Gamma(f(T\sigma)) .$$

The second is covariance under a local modification of the couplings: for $B \subset \mathbb{Z}^d$ finite and $J_B \in \mathbb{R}^{E(B)}$, we define the operation $L_{J_B} : \Gamma \mapsto L_{J_B}\Gamma$ where

$$(6) \quad (L_{J_B}\Gamma)(f(\sigma)) = \frac{\Gamma\left(f(\sigma) \exp\left(-\beta H_{B,J}(\sigma)\right)\right)}{\Gamma\left(\exp\left(-\beta H_{B,J}(\sigma)\right)\right)} .$$

This simply modifies the couplings within a finite subset B of \mathbb{Z}^d . It was shown in [2, 3, 22, 23, 24] that the κ_J 's arising as subsequence limits of finite-volume Gibbs measures are probability measures on Gibbs states. Such measures are referred to as *metastates* [22].

Definition 3.1. *A metastate κ for the EA Hamiltonian on \mathbb{Z}^d is a measurable mapping*

$$(7) \quad \begin{aligned} \mathbb{R}^{E(\mathbb{Z}^d)} &\rightarrow \mathcal{M}_1(\Sigma) \\ J &\mapsto \kappa_J \end{aligned}$$

with the following properties

- (1) **Support on Gibbs states.** *Every state sampled from the κ_J is a Gibbs state for the realization for the couplings. Precisely,*

$$(8) \quad \kappa_J(\mathcal{G}_J) = 1 \text{ } \nu\text{-a.s.}$$

- (2) **Coupling Covariance.** *For $B \subset \mathbb{Z}^d$ finite, $J_B \in \mathbb{R}^{E(B)}$, and any measurable subset A of $\mathcal{M}_1(\Sigma)$,*

$$(9) \quad \kappa_{J+J_B}(A) = \kappa_J(L_{J_B}^{-1}A)$$

where $L_{J_B}^{-1}A = \left\{ \Gamma \in \mathcal{M}_1(\Sigma) : L_{J_B}\Gamma \in A \right\}$.

- (3) **Translation Covariance.** *For any translation T of \mathbb{Z}^d and any measurable subset A of $\mathcal{M}_1(\Sigma)$*

$$(10) \quad \kappa_{TJ}(A) = \kappa_J(T^{-1}A) .$$

The translation covariance is a direct consequence when finite-volume Gibbs measures with periodic boundary conditions are considered. For other coupling-independent boundary conditions, such as free or periodic, it can be recovered by taking an average of the translates of the finite-volume Gibbs measures [2, 4, 25].

4. MAIN RESULTS

Our goal is to understand how the random variable F_Λ fluctuates under the measure M defined in (4). Of course, the appearance of nontrivial fluctuations requires that the states Γ and Γ' differ with positive probability:

$$(11) \quad M\left\{(J, \Gamma, \Gamma') : \Gamma \neq \Gamma'\right\} > 0 .$$

Recall that two elements of $\mathcal{M}_1(\Sigma)$ differ if and only if there exists a correlation function for which the expectations in Γ and Γ' are different. In this paper, we look at the simplest case of (11) where the first moment of the one-edge correlation function differs. This implies the existence of incongruent Gibbs states. Our working assumption will therefore be:

Assumption 4.1. *There exists an edge $(x, y) \in E(\mathbb{Z}^d)$ such that*

$$(12) \quad \nu\left\{J : \kappa_J(\Gamma(\sigma_x \sigma_y)) \neq \kappa'_J(\Gamma'(\sigma'_x \sigma'_y))\right\} > 0 .$$

Note that if the assumption is satisfied for one edge then it is satisfied for all edges, by the translation covariance of the metastates and the translation invariance of the coupling distribution ν . Also, the assumption implies the existence of an incongruent pair (Γ, Γ') in the sense of Definition 1.1: there must exist $\varepsilon > 0$ for which

$$(13) \quad \lim_{\Lambda \rightarrow \mathbb{Z}^d} \frac{1}{|E(\Lambda)|} \sum_{(x, y) \in E(\Lambda)} \kappa_J \times \kappa'_J (|\Gamma(\sigma_x \sigma_y) - \Gamma'(\sigma'_x \sigma'_y)|) > \varepsilon \text{ on a set of } J\text{'s of positive probability.}$$

Thus, $\kappa_J \times \kappa'_J$ must sample incongruent pairs. We remark that the existence of incongruent pairs does not imply Assumption 4.1 since the quantity $\Gamma(\sigma_x \sigma_y) - \Gamma'(\sigma'_x \sigma'_y)$ might fluctuate yet have zero mean when sampled from $\kappa_J \times \kappa'_J$.

Our main result is to prove a lower bound on the fluctuations of F_Λ of the order of square root of the volume in any dimension under this incongruence assumption.

Theorem 4.2. *If Assumption 4.1 holds, then there exists a constant $c > 0$ such that for any $\Lambda = [-L, L]^d \subset \mathbb{Z}^d$ the variance of F_Λ under M satisfies*

$$(14) \quad \text{Var}_M(F_\Lambda) \geq c|\Lambda| .$$

The theorem is proved in Section 5, using a martingale decomposition of F_Λ as discussed in Sec. 2.

Remark. The results of [31] (see in particular Theorem 2.3) imply that the variance of the free energy difference satisfies an *upper* bound of the same form but with a larger constant, as long as the coupling distribution has a finite second moment.

Lower bounds of the type of Theorem 4.2 can be used to rule out certain structures of the metastates and of the underlying Gibbs state in dimension $d = 2$. As an example, we prove the following corollary in Sect. 6.

Corollary 4.3. *In $d = 2$, for every pair of metastates $\kappa.$ and κ' and every edge $(x, y) \in E(\mathbb{Z}^d)$,*

$$(15) \quad \kappa_J(\Gamma(\sigma_x \sigma_y)) = \kappa'_J(\Gamma'(\sigma'_x \sigma'_y)) \quad \nu\text{-a.s.}$$

It is important to remark that Assumption 4.1 can be used to study the structure of a single metastate. If we suppose the metastate is supported on more than one state, then it is possible to condition this measure on two different subsets of incongruent states in order to produce two different measures κ_J and κ'_J . Then we would be in position to use Theorem 4.2 and, in two dimensions, Corollary 4.3. However, the subsets on which we are conditioning must be chosen carefully so that the coupling covariance and the translation covariance of a metastate still hold for the conditional measures. To illustrate this method, we apply it to rule out possible structures of the metastate in dimension two.

Corollary 4.4. *A metastate κ . in $d = 2$ cannot be supported on a countably infinite number of incongruent states, that is it cannot be of the form*

$$\kappa_J(d\Gamma) = \sum_{\alpha \in \mathcal{A}} p_\alpha \delta_{\Gamma^\alpha}$$

for a countable index set \mathcal{A} , $\sum_\alpha p_\alpha = 1$ and with incongruent states $\Gamma^\alpha \in \mathcal{G}_J$ in the sense of Definition 1.1

The important consequence of the countability hypothesis is the existence of two distinct weights $p_\alpha \neq p_{\alpha'}$, which is ensured in the countably infinite case. The proof applies also to the case where \mathcal{A} is finite and the weights are not all equal. Of course, the result is far from the conjecture that in $d = 2$ there exists a single Gibbs state. Our hope is that this method of proof using fluctuation bounds could be extended to further restrict the possible structures of the set of Gibbs states in dimension two and higher.

5. PROOF OF THEOREM 4.2

We start by introducing some notation and reviewing some elementary facts.

For a finite $B \subset \mathbb{Z}^d$, we write $J_B := (J_{xy}, (x, y) \in E(B))$ for the couplings on $E(B)$. We write ∂B for the set of edges that have only one end in B . We denote by J_{B^c} the set of couplings with at least one end outside B ; that is, $J_{B^c} = (J_{xy}, (x, y) \in E(B^c) \cup \partial B)$. For a metastate κ . and a finite set B , we write $\kappa_{J_{B^c}}$ for the metastate conditioned on the realization J but with J_B set to zero.

For two metastates κ . and κ' ., and a finite set $B \subset \mathbb{Z}^d$, we define the measure

$$(16) \quad M_{B^c} := \nu(dJ_{B^c}) \kappa_{J_{B^c}} \times \kappa'_{J_{B^c}} .$$

We shall consider the measure $\nu(dJ_B)M_{B^c}$. The following Lemma proves some important invariance properties of the measure M that will be needed later.

Lemma 5.1 (Invariance properties of M). *Let κ . and κ' . be two metastates and B a finite box in \mathbb{Z}^d . Consider the measures M and M_{B^c} given in (4) and in (16). Then for any measurable function $F : \mathbb{R}^{E(\mathbb{Z}^d)} \times \mathcal{M}_1(\Sigma) \times \mathcal{M}_1(\Sigma) \rightarrow \mathbb{R}$ we have both*

$$(17) \quad F(J, \Gamma, \Gamma') \text{ has the same law under } M \text{ as under } TM \text{ for any translation } T$$

and

$$(18) \quad F(J, L_{J_B}\Gamma, L_{J_B}\Gamma') \text{ has the same law under } \nu(dJ_B)M_{B^c} \text{ as } F(J, \Gamma, \Gamma') \text{ under } M .$$

Proof. The coupling covariance of the metastates implies that, for any Borel measurable function $F : \mathbb{R}^{E(\mathbb{Z}^d)} \times \mathcal{M}_1(\Sigma) \times \mathcal{M}_1(\Sigma) \rightarrow \mathbb{R}$, $F(J, L_{J_B}\Gamma, L_{J_B}\Gamma')$ has the same law under $\nu(dJ_B)M_{B^c}$ as $F(J, \Gamma, \Gamma')$ under M . Furthermore, it is easily checked that translation covariance of the metastate, together with the fact that the distribution on the couplings is itself invariant under translation, gives translation-invariance of the measure: if T is a translation in \mathbb{Z}^d and $F : \mathbb{R}^{E(\mathbb{Z}^d)} \times \mathcal{M}_1(\Sigma) \times \mathcal{M}_1(\Sigma) \rightarrow \mathbb{R}$ is some measurable function, then $TM(F) := M(T^{-1}F) = M(F)$. Here, $M(F) = \int F dM$ is the expectation of F under M . Note that by translation invariance of M , if Assumption 4.1 holds for one edge, it holds for all edges. \square

The next lemma states, without a proof, two elementary properties of the variance that will be needed.

Lemma 5.2. *For X a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub σ -algebra $\mathcal{G} \subset \mathcal{F}$,*

$$(19) \quad \text{Var}X = \mathbb{E}[\text{Var}(X|\mathcal{G})] + \text{Var}(\mathbb{E}[X|\mathcal{G}]) .$$

Also, if X and X' are two independent copies of X , then

$$(20) \quad \text{Var}X = \frac{1}{2}\mathbb{E}[(X - X')^2] .$$

The crucial ingredients of the proof of Theorem 4.2 are the next two lemmas. Recalling the definition of M_{B^c} in (16):

Lemma 5.3. *Consider a measurable function $F : \mathbb{R}^{E(\mathbb{Z}^d)} \times \mathcal{M}_1(\Sigma) \times \mathcal{M}_1(\Sigma) \rightarrow \mathbb{R}$ and a finite subset B of \mathbb{Z}^d . Then*

$$(21) \quad M(F(J, \Gamma, \Gamma')|J_B) = M_{B^c}(F(J, L_{J_B}\Gamma, L_{J_B}\Gamma')) \quad \nu\text{-a.s.}$$

where $M(\cdot|J_B)$ is the conditional expectation given J_B .

Proof. Let $g(J_B)$ be some bounded measurable function of J_B . We need to show that

$$(22) \quad M(F(J, \Gamma, \Gamma')g(J_B)) = \int \nu(dJ_B) M_{B^c}(F(J, L_{J_B}\Gamma, L_{J_B}\Gamma'))g(J_B) .$$

But by Lemma 5.1 the left side equals

$$(23) \quad \int \nu(dJ_B) \int \nu(dJ_{B^c}) \kappa_{J_{B^c}} \times \kappa_{J_{B^c}}(F(J, L_{J_B}\Gamma, L_{J_B}\Gamma')g(J_B))$$

which is, by definition and by taking $g(J_B)$ inside the first expectation,

$$(24) \quad \int \nu(dJ_B) g(J_B) M_{B^c}(F(J, L_{J_B}\Gamma, L_{J_B}\Gamma')) ,$$

as claimed. \square

One consequence of Lemma 5.3 is to provide a smooth version of the conditional expectation of the free energy difference given J_B . In particular, we can compute its derivatives.

Lemma 5.4. *Let F_Λ be as in (3) and $B \subset \Lambda$ finite. Then for any edge (x, y) in B , we have*

$$(25) \quad \frac{\partial}{\partial J_{xy}} M(F_\Lambda|J_B) = \beta M_{B^c}(L_{J_B}\Gamma(\sigma_x\sigma_y) - L_{J_B}\Gamma'(\sigma_x\sigma_y)) \quad \nu\text{-a.s.}$$

Proof. By Lemma 5.3, we need to compute

$$(26) \quad \frac{\partial}{\partial J_{xy}} M_{B^c} \left(F(J, L_{J_B} \Gamma, L_{J_B} \Gamma)' \right) = \frac{\partial}{\partial J_{xy}} M_{B^c} \left(\log L_{J_B} \Gamma \left(\exp \beta H_{J,\Lambda}(\sigma) \right) - \log L_{J_B} \Gamma' \left(\exp \beta H_{J,\Lambda}(\sigma') \right) \right).$$

By (6), the first logarithmic term on the right side is simply

$$(27) \quad \log L_{J_B} \Gamma(\exp \beta H_{J,\Lambda}(\sigma)) = \log \frac{\Gamma(\exp \beta H_{J,\Lambda}(\sigma) \exp -\beta H_{J,B}(\sigma))}{\Gamma(\exp -\beta H_{J,B}(\sigma))}$$

Note that the numerator no longer depends on J_B since its contribution is cancelled. Therefore,

$$(28) \quad \frac{\partial}{\partial J_{xy}} \log L_{J_B} \Gamma(\exp \beta H_{J,\Lambda}(\sigma)) = \beta \frac{\Gamma(\sigma_x \sigma_y \exp -\beta H_{J,B}(\sigma))}{\Gamma(\exp -\beta H_{J,B}(\sigma))} = L_{J_B} \Gamma(\sigma_x \sigma_y).$$

It remains to prove that the derivative can be passed through M_{B^c} . This follows from dominated convergence by noticing that the derivatives are uniformly bounded, completing the proof. \square

With these lemmas in hand, we can now proceed to the proof of the main result of the paper.

Proof of Theorem 4.2. Let $\mathcal{F}_\Lambda = \sigma(J_{x,y}, (x,y) \in E(\Lambda))$. By (19), we have

$$(29) \quad \text{Var}_M(F_\Lambda) \geq \text{Var}_M \left(M(F_\Lambda | \mathcal{F}_\Lambda) \right).$$

Next divide Λ in equally sized blocks B_1, \dots, B_N , with $1 \ll |B_k| \ll |\Lambda|$. The size of the blocks is independent of Λ , so that

$$(30) \quad N = C|\Lambda|$$

for some constant $C > 0$. The size of each B is sufficiently large so that

$$(31) \quad M \left(\Gamma(\sigma_x \sigma_y) - \Gamma'(\sigma_x \sigma_y) | J_B \right) \neq 0 \text{ with positive } \nu\text{-probability.}$$

Indeed, we get by taking $B \rightarrow \mathbb{Z}^d$ that

$$(32) \quad M \left(\Gamma(\sigma_x \sigma_y) - \Gamma'(\sigma_x \sigma_y) | J_B \right) \rightarrow \kappa_J \left(\Gamma(\sigma_x \sigma_y) \right) - \kappa'_J \left(\Gamma(\sigma_x \sigma_y) \right),$$

where we used the Martingale Convergence Theorem for uniformly integrable martingales (see, for example, Theorem 5.5.7 in [11]). Thus, there would be a contradiction with Assumption 4.1 if there were a sequence of volumes $B \rightarrow \mathbb{Z}^d$ such that (31) is zero almost surely along the sequence.

We now consider $\mathcal{F}_k = \sigma(J_{B_i}, i \leq k)$ and the martingale difference $M(F_\Lambda | \mathcal{F}_k) - M(F_\Lambda | \mathcal{F}_{k-1})$. This yields the lower bound

$$(33) \quad \text{Var}_M(F_\Lambda) \geq \sum_{k=1}^N \text{Var}_M \left(M(F_\Lambda | \mathcal{F}_k) - M(F_\Lambda | \mathcal{F}_{k-1}) \right).$$

To prove this inequality we use (19), in addition dropping the couplings between boxes. Again using (19), this time averaging the couplings *outside* a given B_k , gives

$$(34) \quad \text{Var}_M(F_\Lambda) \geq \sum_{k=1}^N \text{Var}_M\left(M(F_\Lambda|J_{B_k})\right).$$

The remainder of the proof focuses on establishing the following two claims:

- (1) $\text{Var}_M\left(M(F_\Lambda|J_{B_k})\right) = \text{Var}_M\left(M(F_\Lambda|J_{B_1})\right)$, so the variance does not depend on k ;
- (2) $\text{Var}_M\left(M(F_\Lambda|J_{B_1})\right) > c'$ for $c' > 0$ independent of Λ .

Once both claims are established, it follows that $\text{Var}_M(F_\Lambda) \geq c'N = c'C|\Lambda|$, thereby concluding the proof.

To prove the first claim, let B be a generic block. Writing the variance using (20) gives

$$(35) \quad \text{Var}_M\left(M(F_\Lambda|J_B)\right) = \frac{1}{2} \int \nu(z_B'') \int \nu(z_B') \left\{ M(F_\Lambda|J_B = z_B') - M(F_\Lambda|J_B = z_B'') \right\}^2$$

By Lemma 5.4, $M(F_\Lambda|J_B)$ is differentiable with respect to J_{xy} for any edge (x, y) in B . In particular, the gradient $\nabla_B M(F_\Lambda|J_B)$ exists almost surely and we can write

$$(36) \quad M(F_\Lambda|J_B = z_B') - M(F_\Lambda|J_B = z_B'') = \int_{z_B' \rightarrow z_B''} \nabla_B M(F_\Lambda|J_B = z_B) \cdot dz_B \quad \nu\text{-a.s.}$$

For an edge (x, y) , define the function

$$(37) \quad \delta_{xy}(\Gamma, \Gamma') = \Gamma(\sigma_x \sigma_y) - \Gamma'(\sigma'_x \sigma'_y) .$$

By Lemma 5.4, we have

$$(38) \quad \frac{\partial}{\partial J_{xy}} M(F_\Lambda|J_B) = \beta M_{B^c}(\delta_{xy}(L_{J_B} \Gamma, L_{J_B} \Gamma')) .$$

Lemma 5.3 applied to the function δ_{xy} implies that

$$(39) \quad M_{B^c}(\delta_{xy}(L_{J_B} \Gamma, L_{J_B} \Gamma')) = M(\delta_{xy}(\Gamma, \Gamma')|J_B) \quad \nu\text{-a.s.}$$

Writing $M(\delta_B(\Gamma, \Gamma')|J_B) := (M(\delta_{xy}(\Gamma, \Gamma')|J_B), (x, y) \in E(B))$, we have

$$(40) \quad \text{Var}_M\left(M(F_\Lambda|J_B)\right) = \frac{\beta^2}{2} \int \nu(z_B'') \int \nu(z_B') \left\{ \int_{z_B' \rightarrow z_B''} M(\delta_B(\Gamma, \Gamma')|J_B = z_B) \cdot dz_B \right\}^2$$

For a translation T such that $TB \subset \Lambda$, consider $\text{Var}_M(M(F_\Lambda|J_{TB}))$. Note that

$$(41) \quad \delta_{TxTy}(\Gamma, \Gamma') = \Gamma(\sigma_{Tx} \sigma_{Ty}) - \Gamma'(\sigma'_{Tx} \sigma'_{Ty}) = T\Gamma(\sigma_x \sigma_y) - T\Gamma'(\sigma'_x \sigma'_y) .$$

In particular, $\delta_{TxTy}(\Gamma, \Gamma')$ has the same law under M as $\delta_{xy}(\Gamma, \Gamma')$ by Lemma 5.1. We conclude that

$$(42) \quad \text{Var}_M\left(M(F_\Lambda|J_{TB})\right) = \text{Var}_M\left(M(F_\Lambda|J_B)\right) .$$

This proves Claim 1.

To prove Claim 2 we observe that, since the gradient does not depend on Λ , we simply need to show that

$$(43) \quad \int \nu(z_B'') \int \nu(z_B') \left\{ \int_{z_B' \rightarrow z_B''} M(\delta_B(\Gamma, \Gamma') | J_B = z_B) \cdot dz_B \right\}^2 > 0 .$$

Suppose the contrary. Then

$$(44) \quad \int_{z_B' \rightarrow z_B''} M(\delta_B(\Gamma, \Gamma') | J_B = z_B) \cdot dz_B = 0 \quad \text{for } \nu\text{-almost all } (z_B', z_B'').$$

In particular, since ν is continuous, this implies

$$(45) \quad M(\delta_B(\Gamma, \Gamma') | J_B) = 0 \quad \nu\text{-a.s.}$$

In other words, for any edge $(x, y) \in E(B)$,

$$(46) \quad M(\Gamma(\sigma_x \sigma_y) - \Gamma'(\sigma_x \sigma_y) | J_B) = 0 \quad \nu\text{-a.s.}$$

which contradicts (31). □

6. PROOF OF THE RESULTS IN $d = 2$

The proof of Corollary 4.3 proceeds by contradiction, studying the fluctuations of $M(F_\Lambda | J_\Lambda)$ in the spirit of [2].

The outline of the proof is as follows. We always have by Proposition 6.1 below that

$$(47) \quad |M(F_\Lambda | J_\Lambda)| \leq 4\beta \sum_{e \in \partial\Lambda} \nu(|J_e|) = 4\beta |\partial\Lambda| \nu(|J_e|) .$$

In particular, for all $t > 0$

$$(48) \quad \nu\left(\exp t \frac{M(F_\Lambda | J_\Lambda)}{|\partial\Lambda|}\right) \leq e^{4\beta t} .$$

To get a contradiction, we would like to show that, under Assumption 4.1, the distribution of

$$(49) \quad \frac{M(F_\Lambda | J_\Lambda)}{|\partial\Lambda|}$$

has at least a Gaussian tail, in the sense that there exists $c > 0$ such that

$$(50) \quad \nu\left(\exp t \frac{M(F_\Lambda | J_\Lambda)}{|\partial\Lambda|}\right) \geq e^{ct^2} .$$

This is possible in $d = 2$ where, unlike in higher dimensions, $|\partial\Lambda|$ is of the order of the square root of $|E(\Lambda)|$. This is not true in higher dimensions. Eqs. (48) and (50) are in obvious contradiction for t large enough.

6.1. Upper bound for the difference of free energies.

Proposition 6.1. *For F_Λ defined in (3), we have*

$$(51) \quad \left| F_\Lambda(J, \Gamma, \Gamma') \right| \leq 4\beta \sum_{e \in \partial\Lambda} |J_e|, \quad M\text{-a.s.}$$

In particular,

$$(52) \quad \limsup_{\Lambda \rightarrow \mathbb{Z}^d} \frac{1}{|\partial\Lambda|} \left| F_\Lambda(J, \Gamma, \Gamma') \right| \leq 4\beta \nu(|J_e|) \quad M\text{-a.s.}$$

Write $G_{\beta, \Lambda, J}$ for the Gibbs measure in a finite box Λ . (Recall that the Hamiltonian $H_{\Lambda, J}$ does not include interaction terms between Λ and the boundary $\partial\Lambda$).

$$(53) \quad G_{\beta, \Lambda, J}(f(\sigma_\Lambda)) = \frac{\sum_{\sigma_\Lambda \in \{-1, +1\}^\Lambda} f(\sigma_\Lambda) \exp(-\beta H_{\Lambda, J}(\sigma))}{\sum_{\sigma_\Lambda \in \{-1, +1\}^\Lambda} \exp(-\beta H_{\Lambda, J}(\sigma))}$$

To prove the Proposition, we need the following lemma.

Lemma 6.2. *Let $\Gamma \in \mathcal{G}_{\beta, J}$ be a Gibbs state and Λ a finite box of \mathbb{Z}^d . For $f : \{-1, 1\}^\Lambda \rightarrow [0, \infty)$ a positive measurable function, we have*

$$(54) \quad \exp\left(-2\beta \sum_{e \in \partial\Lambda} |J_e|\right) \leq \frac{\Gamma(f(\sigma_\Lambda))}{G_{\beta, \Lambda, J}(f(\sigma_\Lambda))} \leq \exp\left(2\beta \sum_{e \in \partial\Lambda} |J_e|\right)$$

Proof. Consider the Hamiltonian of interactions between spins in Λ and spins in Λ^c :

$$(55) \quad H_{\partial\Lambda, J}(\sigma) = \sum_{(x, y) \in \partial\Lambda} -J_{xy} \sigma_x \sigma_y.$$

Clearly,

$$(56) \quad \exp\left(-\beta \sum_{e \in \partial\Lambda} |J_e|\right) \leq \exp(-\beta H_{\partial\Lambda, J}(\sigma)) \leq \exp\left(\beta \sum_{e \in \partial\Lambda} |J_e|\right)$$

Since Γ is a Gibbs state the DLR equations imply

$$(57) \quad \Gamma(f(\sigma_\Lambda)) = \Gamma\left(\frac{\sum_{\sigma_\Lambda \in \{-1, +1\}^\Lambda} f(\sigma_\Lambda) \exp(-\beta(H_{\Lambda, J}(\sigma) + H_{\partial\Lambda, J}(\sigma)))}{\sum_{\sigma_\Lambda \in \{-1, +1\}^\Lambda} \exp(-\beta(H_{\Lambda, J}(\sigma) + H_{\partial\Lambda, J}(\sigma)))}\right).$$

Since f is assumed positive, the bounds (56) give the desired result. \square

Proof of Proposition 6.1. A direct application of Lemma 6.2 with $f(\sigma_\Lambda) = \exp \beta H_{\Lambda, J}(\sigma)$ produces the bounds

$$(58) \quad -4\beta \sum_{e \in \partial\Lambda} |J_e| \leq \log \frac{\Gamma(\exp \beta H_{\Lambda, J}(\sigma))}{\Gamma'(\exp \beta H_{\Lambda, J}(\sigma))} \leq 4\beta \sum_{e \in \partial\Lambda} |J_e|.$$

Therefore

$$(59) \quad \left| \frac{F_\Lambda(J, \Gamma, \Gamma')}{|\partial\Lambda|} \right| \leq \frac{4\beta}{|\partial\Lambda|} \sum_{e \in \partial\Lambda} |J_e|.$$

The conclusion follows from the strong law of large numbers. \square

6.2. Proof of Corollary 4.3. To prove (50), we need to improve Theorem 4.2 from a lower bound on the variance to a lower bound on the moment generating function. For this, we use the following proposition which is a version of Proposition A.2.1 in [2]. It is a weaker result than the central limit theorem for martingale arrays. It provides a lower bound on the moment generating function of the martingale whenever a lower bound on the quadratic variation (in the probabilistic sense) is available.

Proposition 6.3 (Proposition A.2.1 in [2]). *For every $n \in \mathbb{N}$, let $(X_{n,k}, k \leq n)$ be a martingale with respect to the filtration $(\mathcal{F}_{n,k}, k \leq n)$. Suppose that the martingale differences $\Delta X_{n,k} := X_{n,k} - X_{n,k-1}$ satisfy the two conditions below*

(1) *There exists σ^2 such that*

$$(60) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left(\sum_{k=1}^n \mathbb{E}[\Delta X_{n,k}^2 | \mathcal{F}_{n,k-1}] \leq \sigma^2 - \delta \right) \rightarrow 0 \text{ for every } \delta > 0 .$$

(2) *For every $\delta > 0$,*

$$(61) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n \mathbb{E} \left[\Delta X_{n,k}^2 1_{\{|\Delta X_{n,k}| > \delta\}} \right] = 0 .$$

Then

$$(62) \quad \liminf_{n \rightarrow \infty} \mathbb{E}[\exp t X_{n,n}] \geq e^{\frac{t^2 \sigma^2}{2}} .$$

The proposition will be used with the following quantities. Consider the lexicographic order on \mathbb{Z}^2 : $(x_1, x_2) \preceq (y_1, y_2)$ iff $x_1 < y_1$ or $x_1 = y_1$ and $x_2 \leq y_2$. It induces an order on $E(\mathbb{Z}^2)$ by enumerating the edges in order of the vertices at which they originate, with (say) the edge on top of a vertex preceding the edge on its right. We will use the same notation for both orders and thus write $e \preceq e'$ if e precedes e' in the above sense on $E(\mathbb{Z}^2)$.

Define the following σ -algebras

$$(63) \quad \begin{aligned} \mathcal{F}_{\preceq e} &:= \sigma \left(J_{xy} : (x, y) \preceq e \right) \\ \mathcal{F}_\Lambda &:= \sigma \left(J_{xy} : (x, y) \in E(\Lambda) \right) \\ \mathcal{F}_{\Lambda, \preceq e} &:= \mathcal{F}_\Lambda \cap \mathcal{F}_{\preceq e} \end{aligned}$$

It will be convenient to enumerate the edges in $E(\Lambda)$ in the above order: $E(\Lambda) = \{e_1, e_2, \dots, e_k, \dots, e_{|E(\Lambda)|}\}$. We define

$$(64) \quad \begin{aligned} Y_{\Lambda, k} &:= M \left(F_\Lambda | \mathcal{F}_{\Lambda, \preceq e_k} \right) \\ \Delta Y_{\Lambda, k} &:= Y_{\Lambda, k} - Y_{\Lambda, k-1} . \end{aligned}$$

Plainly, for each Λ , $(Y_{\Lambda, k}, k \leq |E(\Lambda)|)$ is a martingale for the filtration $(\mathcal{F}_{\Lambda, \preceq e_k}, k \leq |E(\Lambda)|)$. Note that with this notation $\sum_{k=1}^{|E(\Lambda)|} \Delta Y_{\Lambda, k} = M(F_\Lambda | \mathcal{F}_\Lambda) - M(F_\Lambda)$. Equation (50) will be proved by showing that

$$(65) \quad \left(\frac{Y_{\Lambda, k}}{\sqrt{|E(\Lambda)|}}, k \leq |E(\Lambda)| \right)$$

satisfies the two hypotheses of Proposition 6.3. In what follows it will be convenient to use the notation $J_{\prec e}$ to denote $(J_{e'}, e' \prec e)$ and similarly for $J_{\succ e}$. We can define using this notation

$$(66) \quad \overline{F}_\Lambda(J) = \overline{F}_\Lambda(J_{\prec e}, J_e, J_{\succ e}) := M(F_\Lambda | \mathcal{F}_\Lambda) .$$

We start with the simpler part.

Proof of Hypothesis 2). Observe that, by Lemma 5.4, $\Delta Y_{\Lambda,k}$ can be represented as

$$(67) \quad \begin{aligned} \Delta Y_{\Lambda,k}(J_{e_k}, J_{\prec e_k}) &= \int \nu(dJ_{\Lambda^c}) \int \nu(dJ_{\succ e_k}) \left\{ \overline{F}_\Lambda(J_{\prec e_k}, J_{e_k}, J_{\succ e_k}) - \int \nu(dy) \overline{F}_\Lambda(J_{\prec e_k}, y, J_{\succ e_k}) \right\} \\ &= \int \nu(dJ_{\Lambda^c}) \int \nu(dJ_{\succ e_k}) \int \nu(dy) \left\{ \int_y^{J_{e_k}} D_k(J_{\prec e_k}, s, J_{\succ e_k}) ds \right\} \end{aligned}$$

where for $e_k = (x, y)$ we defined $D_k(J) = \beta \left\{ \kappa_J(\Gamma(\sigma_x \sigma_y)) - \kappa'_J(\Gamma'(\sigma_x \sigma_y)) \right\}$. In particular,

$$(68) \quad |\Delta Y_{\Lambda,k}(J_{e_k}, J_{\prec e_k})| \leq 2\beta \int \nu(dy) |J_{e_k} - y| \leq 2\beta(|J_{e_k}| + \nu(|J_e|)) ,$$

and

$$(69) \quad \nu\left(\Delta Y_{\Lambda,k}^2\right) \leq 16\beta^2 \nu(J_e^2) ,$$

for all $k \leq |E(\Lambda)|$. Also $\nu\left(\Delta Y_{\Lambda,k}^4\right) < \infty$ uniformly in k if we suppose that $\nu(J_e^4) < \infty$.

For $\delta > 0$, the above implies that

$$(70) \quad \nu\left\{ |\Delta Y_{\Lambda,k}| > \delta \sqrt{|E(\Lambda)|} \right\} \leq \frac{\nu\left(\Delta Y_{\Lambda,k}^2\right)}{\delta^2 |E(\Lambda)|} \rightarrow 0 \text{ as } \Lambda \rightarrow \mathbb{Z}^d, \text{ uniformly in } k.$$

Hence, by Cauchy's inequality, for some constant $C > 0$

$$(71) \quad \sum_{k=1}^{|E(\Lambda)|} \nu\left(\frac{\Delta Y_{\Lambda,k}^2}{|E(\Lambda)|} 1_{\{|\Delta Y_{\Lambda,k}| > \delta |E(\Lambda)|^{1/2}\}}\right) \leq C \frac{\max_k \nu\left(\Delta Y_{\Lambda,k}^4\right)^{1/2} \nu(J_e^2)^{1/2}}{\delta |E(\Lambda)|^{1/2}} \rightarrow 0 .$$

□

Proof of Hypothesis 1). We show that

$$(72) \quad \lim_{\Lambda \rightarrow \mathbb{Z}^d} \frac{1}{|E(\Lambda)|} \sum_{k=1}^{|E(\Lambda)|} \nu(\Delta Y_{\Lambda,k}^2 | \mathcal{F}_{\Lambda, \preceq e_{k-1}}) = \sigma^2$$

where the convergence holds in $L^1(\nu)$ (hence also in ν -probability) and σ^2 is a constant. Note that, under Assumption 4.1, σ^2 is non-zero. Indeed, we have by conditioning

$$(73) \quad \frac{1}{|E(\Lambda)|} \text{Var}\left(M(F_\Lambda | \mathcal{F}_\Lambda)\right) = \frac{1}{|E(\Lambda)|} \text{Var}\left(\sum_{k=1}^{|E(\Lambda)|} \Delta Y_{\Lambda,k}\right) = \frac{1}{|E(\Lambda)|} \sum_{k=1}^{|E(\Lambda)|} \nu\left(\Delta Y_{\Lambda,k}^2\right) .$$

The right side converges to σ^2 by taking the expectation in (72) (L^1 -convergence implies convergence of the expectations). On the other hand, the proof of Theorem 4.2 shows that the left side is strictly greater than 0 uniformly in Λ .

The proof is based on the L^2 -ergodic theorem. To use it, we need to work around the dependence on Λ . For this purpose, define the following quantities,

$$(74) \quad \begin{aligned} \tilde{Y}_{\Lambda,k} &:= M\left(F_\Lambda | \mathcal{F}_{\preceq e_k}\right) \\ \Delta \tilde{Y}_k &:= Y_k - Y_{k-1} \end{aligned}$$

Note that $Y_{\Lambda,k} = \nu(\tilde{Y}_{\Lambda,k} | \mathcal{F}_\Lambda)$. Similarly as in (67)

$$(75) \quad \Delta \tilde{Y}_k = \int \nu(dJ_{\succ e_k}) \int \nu(dy) \left\{ \int_y^{J_{e_k}} D_k(J_{\preceq e_k}, s, J_{\succ e_k}) ds \right\}$$

where for $e_k = (x, y)$ we defined $D_k(J) = \beta \left\{ \kappa_J(\Gamma(\sigma_x \sigma_y)) - \kappa'_J(\Gamma'(\sigma_x \sigma_y)) \right\}$. In particular, $\Delta \tilde{Y}_k$ does not depend on Λ (as the notation suggests). Moreover, it is easily checked that $\nu(\Delta \tilde{Y}_k^4) < \infty$.

The crucial point is the fact that the random variables $f_{e_k}(J) := \nu(\Delta \tilde{Y}_k^2 | \mathcal{F}_{\preceq e_{k-1}})$ are translates, that is if T is a translation mapping e_k to e_l then

$$(76) \quad f_{e_l}(J) = f_{e_k}(T^{-1}J) .$$

(Note that horizontal and vertical edges are not translates. Thus, the functions $f_{e_k}(J)$ can be reduced to two functions up to translation, not one.) We can therefore apply von Neumann's ergodic theorem (see e.g. [30]) to conclude that

$$(77) \quad \frac{1}{|E(\Lambda)|} \sum_{k=1}^{|E(\Lambda)|} \nu(\Delta \tilde{Y}_k^2 | \mathcal{F}_{\preceq e_{k-1}})$$

converges in $L^2(\nu)$ to a constant, say σ^2 .

To prove (72), it remains to show that we can replace $\nu(\Delta \tilde{Y}_k^2 | \mathcal{F}_{\preceq e_{k-1}})$ by $\nu(\Delta \tilde{Y}_k^2 | \mathcal{F}_{\Lambda, \preceq e_{k-1}})$ and $\nu(\Delta \tilde{Y}_k^2 | \mathcal{F}_{\Lambda, \preceq e_{k-1}})$ by $\nu(\Delta Y_{\Lambda,k}^2 | \mathcal{F}_{\Lambda, \preceq e_{k-1}})$ in (77). In other words, we need to show that we do not lose much by averaging the couplings outside Λ . Precisely, we show, as $\Lambda \rightarrow \mathbb{Z}^2$

$$(78) \quad \nu\left(\left|\nu(\Delta \tilde{Y}_k^2 | \mathcal{F}_{\Lambda, \preceq e_{k-1}}) - \nu(\Delta Y_{\Lambda,k}^2 | \mathcal{F}_{\Lambda, \preceq e_{k-1}})\right|\right) \rightarrow 0$$

$$(79) \quad \nu\left(\left|\nu(\Delta \tilde{Y}_k^2 | \mathcal{F}_{\preceq e_{k-1}}) - \nu(\Delta \tilde{Y}_{\Lambda,k}^2 | \mathcal{F}_{\Lambda, \preceq e_{k-1}})\right|\right) \rightarrow 0$$

uniformly in k . (In fact, as it will be clear from our reasoning, the convergence is uniform for e_k not too close to the boundary of Λ .) The convergence is in L^1 and enough for our purpose.

We first show (78). By Jensen's inequality, the left-hand side is smaller than

$$(80) \quad \nu\left(\left|\Delta \tilde{Y}_k^2 - \Delta Y_{\Lambda,k}^2\right|\right)$$

Factoring the difference of the squares and using the Cauchy-Schwarz inequality, we get the upper bound

$$(81) \quad \nu\left(\left|\Delta\tilde{Y}_k - \Delta Y_{\Lambda,k}\right|^2\right)^{1/2} \nu\left(\left|\Delta\tilde{Y}_k + \Delta Y_{\Lambda,k}\right|^2\right)^{1/2}$$

The second term is finite, uniformly in k , as a consequence of triangle's inequality and equations (67) and (75). It remains to prove that $\nu\left(\left|\Delta\tilde{Y}_k - \Delta Y_{\Lambda,k}\right|^2\right)$ goes to 0 uniformly in k . Again, by Jensen's inequality and the representations (67) and (75), we have

$$(82) \quad \nu\left(\left|\Delta\tilde{Y}_k - \Delta Y_{\Lambda,k}\right|^2\right) \leq \nu\left(\left|W_k - \nu(W_k|\mathcal{F}_\Lambda)\right|^2\right)$$

where

$$(83) \quad W_k = W_k(J_{\prec e_k}, J_{\succ e_k}) := \int \nu(dy) \left\{ \int_y^{J_{e_k}} D_k(J_{\prec e_k}, s, J_{\succ e_k}) ds \right\}.$$

Note that all W_k 's have the same L^2 -norm since the function is translation-invariant. However, the e_k 's do not have the same location in the box Λ . Thus, we cannot conclude right away, using the L^2 -martingale convergence theorem, that the term converges to 0 uniformly in k . To get around this, note that $\nu\left(\left|W_k - \nu(W_k|\mathcal{F}_\Lambda)\right|^2\right)$ decreases as Λ grows. Therefore, it suffices to take the conditioning on a smaller box than Λ . For example, take each e_k that is at least $|\Lambda|^{1/4}$ (say) away from the boundary (there are $(1+o(1))|E(\Lambda)|$ such edges), and consider a box B_k of area $|\Lambda|^{1/2}$ centered at e_k . Then, by the previous remark,

$$(84) \quad \nu\left(\left|W_k - \nu(W_k|\mathcal{F}_\Lambda)\right|^2\right) \leq \nu\left(\left|W_k - \nu(W_k|\mathcal{F}_{B_k})\right|^2\right)$$

where \mathcal{F}_{B_k} is a box of area $|\Lambda|^{1/2}$ centered at e_k . The right side is now independent on k and goes to 0 as $\Lambda \rightarrow \mathbb{Z}^2$ by the L^2 -martingale convergence theorem.

For (79), the term is smaller than

$$(85) \quad \nu\left(\left|\Delta\tilde{Y}_k^2 - \nu(\Delta\tilde{Y}_{\Lambda,k}^2|\mathcal{F}_\Lambda)\right|\right).$$

With the notation introduced above, we can write

$$(86) \quad \Delta\tilde{Y}_k = \nu(W_k|\mathcal{F}_{\prec e_k}).$$

Again, this quantity does not depend on k . Using the same reasoning as above by reducing \mathcal{F}_Λ to \mathcal{F}_{B_k} , we deduce the convergence to 0 uniformly in k for e_k at least $|\Lambda|^{1/4}$ away from the boundary. \square

7. PROOF OF COROLLARY 4.4

The idea of the proof is to use the weights in the decomposition of the metastate as a *tag* to distinguish the states as J is varied.

Suppose that the metastate κ is of the form $\sum_{\alpha \in \mathcal{A}} p_\alpha \delta_{\Gamma^\alpha}$. First, we observe that the set of weight $(p_\alpha, \alpha \in \mathcal{A})$ does not depend on J . Indeed, by the translation covariance of the metastate $\kappa_{TJ}(d\Gamma) = \kappa_J(dT\Gamma)$. In particular, the weight associated with Γ_α is the same as the weight of $T\Gamma_\alpha$. This implies that the set of weights is translation-invariant as a function of J . Thus, it must be constant ν -almost surely. By hypothesis, there must exist α, α' such

that $p_\alpha \neq p_{\alpha'}$. Let Γ^α and $\Gamma^{\alpha'}$ be the corresponding states. These states belong to \mathcal{G}_J and we write $\Gamma^\alpha = \Gamma_J^\alpha$ and $\Gamma^{\alpha'} = \Gamma_J^{\alpha'}$ to make the dependence on J explicit. Therefore each tag yields a well-defined measurable map of the couplings to $M_1(\Sigma)$

$$(87) \quad J \mapsto \kappa_J^\alpha := \delta_{\Gamma_J^\alpha} \quad J \mapsto \kappa_J^{\alpha'} := \delta_{\Gamma_J^{\alpha'}} .$$

(If more than one state has weight p_α , we can take Γ^α to be the weighted linear combination of these states without loss of generality.)

To prove the corollary, we show that κ_J^α and $\kappa_J^{\alpha'}$ satisfy the properties of a metastate in Definition 3.1 and also satisfies Assumption 4.1. This is in contradiction with Corollary 4.3. The property of support of metastates on Gibbs states is obvious. The translation covariance is also clear since the weight p_α puts Γ_J^α and $T\Gamma_J^\alpha$ in bijection by the translation covariance of the original metastate κ . For the same reason, Γ_J^α is mapped to $L_{J_B}\Gamma_J^\alpha$ when we consider κ_{J+J_B} . This ensures coupling covariance. Therefore, Corollary 4.3 holds. On the other hand, by hypothesis, Γ^α and $\Gamma^{\alpha'}$ are incongruent, thus by Definition 1.1, there exists $\varepsilon > 0$ such that

$$(88) \quad \liminf_{\Lambda \rightarrow \mathbb{Z}^2} \frac{1}{|E(\Lambda)|} \sum_{(x,y) \in E(\Lambda)} 1_{\{|\Gamma_J^\alpha(\sigma_x \sigma_y) - \Gamma_J^{\alpha'}(\sigma_x \sigma_y)| > \varepsilon\}} > 0$$

The left-hand side is a translation-invariant function of the J since $\Gamma_{TJ}^\alpha = T\Gamma_J^\alpha$. In particular, the limit in (88) exists and equals

$$(89) \quad \nu \left\{ J : |\Gamma_J^\alpha(\sigma_x \sigma_y) - \Gamma_J^{\alpha'}(\sigma_x \sigma_y)| > \varepsilon \right\}$$

In particular, Assumption 4.1 is fulfilled. This concludes the proof of the corollary.

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